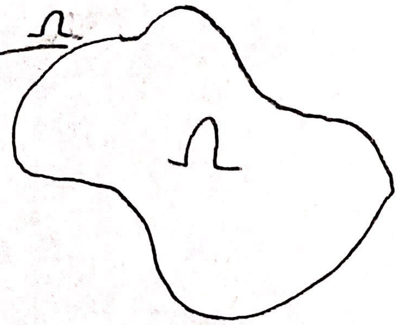
5/26/20

Scattering Theory: 1d Meromorphic Continuation

Study of Open Systems

Closed system example: Wave eq. on Ω



$\Omega \subseteq \mathbb{R}^n$ bounded

$(\partial_t^2 + \Delta)u = 0$ $\quad$ ($\Delta = -\sum_j \partial_{x_j}^2$ convention)

$u|_{\mathbb{R}_+ \times \partial\Omega} = 0$

$u|_{t=0} = f_0 \quad \partial_t u|_{t=0} = f_1$

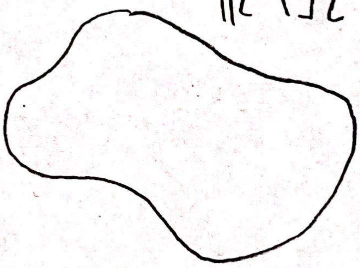
Solution: Δ_x has compact inverse, so discrete spectrum $\lambda_1^2, \lambda_2^2, \ldots \to \infty$

ONB of eigenfunctions $u_1, u_2, \ldots$

Then:

$$\underbrace{u(t,x) = \sum_j a_j e^{-it\lambda_j} u_j(x)}_{\text{eigenfunction expansion}}$$

Open system example: Wave eq. on $\mathbb{R}^n \setminus \Omega$



$(\partial_t^2 + \Delta)u = 0$

$u|_{\mathbb{R}_+ \times \partial\Omega} = 0$

$u|_{t=0} = f_0, \quad \partial_t u|_{t=0} = f_1 \qquad f_0, f_1 \in C_0^\infty(\mathbb{R}^n)$

No eigenvalues this time!

Goal: "Expand" as

$$w(t,x) \sim \sum_j e^{-it\lambda_j} v_j(x) \quad \text{as } t \to \infty \text{ on compact sets}$$

Here $\lambda_j \in \mathbb{C}$ are resonances

$\operatorname{Re} \lambda_j$ gives rate of oscillation

$-\operatorname{Im} \lambda_j$ gives rate of decay

Consider 1d wave equation with potential

$P_V = D_x^2 + V(x)$ where $D_x = \frac{1}{i}\partial_x$, $V \in L^\infty_0(\mathbb{R})$

Wave Equation: $\boxed{\begin{aligned}(-\partial_t^2 - P_V)v &= F \\ v|_{t=0} &= v_0 \\ \partial_t v|_{t=0} &= v_1\end{aligned}}$ $F \in L^1_{loc}(\mathbb{R}; L^2(\mathbb{R}))$

$v_0, v_1 \in L^2(\mathbb{R})$

Stationary wave equation is

$\boxed{(P_V - \lambda^2)u = f}$ $f \in L^2_0(\mathbb{R})$. Convention: $\operatorname{Arg}\lambda \in [0, \pi)$

Note: We do not ask that $u \in L^2(\mathbb{R})$.

Let $\operatorname{supp} V, \operatorname{supp} f \in (-R, R)$

For $\pm x > R$:

$u(x) = a_\pm e^{i\lambda|x|} + b_\pm e^{-i\lambda|x|}$

Outgoing Solutions

$u(x) = a_- e^{-i\lambda x}$ for $x < -R$

$= a_+ e^{i\lambda x}$ for $x > R$

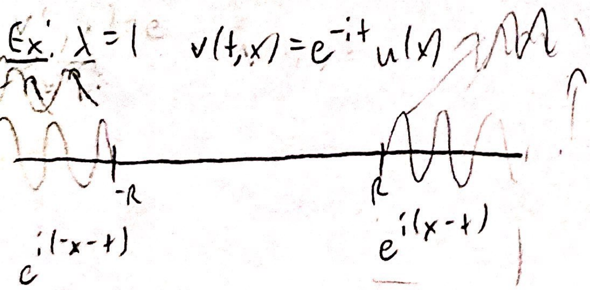
Incoming Solutions

$u(x) = b_- e^{i\lambda x}$ for $x < -R$

$= b_+ e^{-i\lambda x}$ for $x > R$

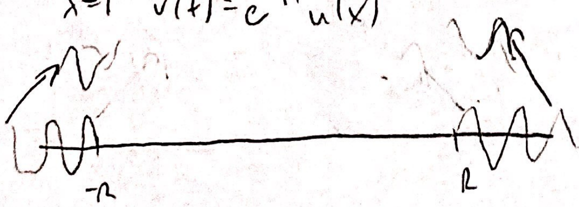
Ex: $\lambda = 1 + i$

Ex: $\lambda = 1$ $v(t,x) = e^{-it}u(x)$



Explanation of names "outgoing" & "incoming"

$\lambda = 1$ $v(t) = e^{-it}u(x)$



Note: If $\operatorname{Im}\lambda > 0$, u outgoing, then $u \in L^2(\mathbb{R})$

If $f = 0$, λ is an actual eigenvalue of P_V.

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Motivation (Not rigorous)

Recall we want to solve

$$(-\partial_t^2 - P_V)v = F$$

Fourier transform in time (actually IFT) ^

$$(P_V - \lambda^2)u(x,\lambda) = f(x,\lambda) \quad \text{where } u = \check{v} \quad f = \check{F}$$

"Invert" $P_V - \lambda^2$ with $R_V(\lambda)$ (called resolvent)

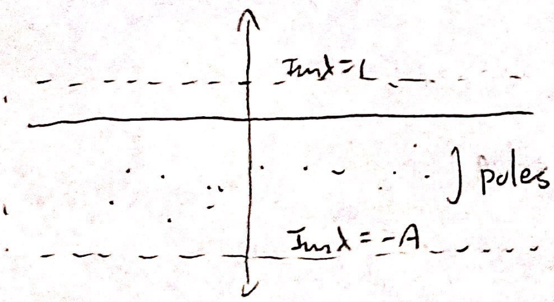
$$u(x,\lambda) = R_V(\lambda) f(x,\lambda)$$

Use meromorphic continuation to extend u to all $\lambda \in \mathbb{C}$

Then $v(t,x) = \frac{1}{2\pi}\int_{\operatorname{Im}\lambda = 1} e^{-it\lambda} u(\lambda, x)$ (Fourier inversion on $e^{-t}v(t,x)$)

$$= \frac{1}{2\pi}\int_{\operatorname{Im}\lambda=1} e^{-it\lambda} R_V(\lambda) f(\lambda)\,d\lambda$$

$$= \frac{1}{2\pi}\int_{\operatorname{Im}\lambda = -A} e^{-it\lambda} R_V(\lambda) f(\lambda)\,d\lambda + \underbrace{\sum_{\substack{\lambda_j \text{ pole} \\ 1 > \operatorname{Im}\lambda_j > -A}} 2\pi i \operatorname{Res}_{\lambda_j}\left(e^{-it\lambda} R_V(\lambda) f(\lambda)\right)}_{\text{This is resonance expansion}}$$



But... None of this was rigorous, because we don't know how to extend $R(\lambda)$ meromorphically!

That will be the subject of the rest of the talk.

V=0 case (The free resolvent)

Want to invert $D_x^2 - \lambda^2$

Use Fourier methods

$$(D_x^2-\lambda^2)u = f \xLeftrightarrow{\text{Fourier}} (\xi^2-\lambda^2)\hat{u} = \hat{f}$$
$$\hat{u} = \frac{1}{\xi^2-\lambda^2}\hat{f}$$
$$u(x) = \frac{i}{2\lambda}\int e^{i\lambda|x-y|}f(y)\,dy \qquad (\text{for } \operatorname{Im}\lambda>0).$$

$R_0(\lambda,x,y) = \frac{i}{2\lambda}e^{i\lambda|x-y|}$ for $\operatorname{Im}\lambda>0$

This has meromorphic continuation to $\mathbb{C}$, with pole at 0.

(<u>Note</u>: Meromorphic continuation is <u>not</u> same as L^2 inverse: for $\operatorname{Im}\lambda<0$, $R_0(-\lambda)$ inverts $D_x^2-\lambda^2$)

<u>Remark</u>: spec D_x^2 is $[0,\infty)$ by applying Fourier transform.

* <u>Thm</u> (<u>Meromorphic Continuation of Free Resolvent</u>):

The free resolvent R_0 extends to meromorphic family of operators

$R_0(\lambda): L^2_0 \to L^2_{loc}$ with $\lambda\in\mathbb{C}$,

$\boxed{\|R_0(\lambda)\|_{L^2\to L^2} = \frac{1}{d(\lambda^2,[0,\infty))} \le \frac{1}{|\lambda|\operatorname{Im}\lambda} \quad \text{for } \operatorname{Im}\lambda>0}$ (In fact, $\|R_0(\lambda)\|_{L^2\to H^j} \le \frac{\langle\lambda\rangle^j}{|\lambda|\operatorname{Im}\lambda}$ for $j\in[0,2]$)

and for $\rho\in C_0^\infty(-L,L)$:

$\boxed{\|\rho R_0(\lambda)\rho\|_{L^2\to H^j} \lesssim e^{2L(\operatorname{Im}\lambda)_-}|\lambda|^{-1}\langle\lambda\rangle^j}$ where $j\in[0,2]$ (here $x_- = \max(0,-x)$, $\langle x\rangle = \sqrt{1+|x|^2}$)

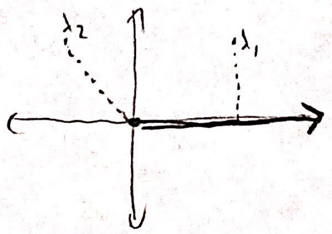
<u>Proof</u>: Extend meromorphically with Schwartz kernel

$$R_0(\lambda,x,y) = \frac{i}{2\lambda}e^{i\lambda|x-y|},$$

<u>For $\operatorname{Im}\lambda>0$</u>:

$$\|R_0(\lambda)u\|_{L^2} = \left\|\frac{1}{\xi^2-\lambda^2}\hat{u}\right\|_{L^2} \quad (\text{Plancherel})$$
$$\le \frac{1}{d(\lambda^2,[0,\infty))}\|u\|_{L^2}$$

Distances to $[0,\infty)$



<u>For general λ</u>:

Use Schur test: $\int|\rho(x)\rho(y)R_0(\lambda,x,y)|dx = |\lambda|^{-1}\int|\rho(x)\rho(y)|e^{-\operatorname{Im}\lambda|x-y|}dx$
$\le |\lambda|^{-1}e^{2L(\operatorname{Im}\lambda)_-}$

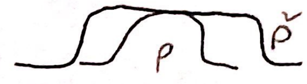
Similarly $\int|\rho(x)\rho(y)R_0(\lambda,x,y)|dy \le |\lambda|^{-1}e^{2L(\operatorname{Im}\lambda)_-}$

By Schur's test $\|R_0(\lambda)\|_{L^2\to L^2} \le |\lambda|^{-1}e^{2L(\operatorname{Im}\lambda)_-}$

For $j=2$, use <u>elliptic regularity</u>

$$\|u\|_{H^2(U)} \lesssim \|u\|_{L^2(W)} + \|D_x^2 u\|_{L^2(W)} \quad \text{for } U \subset\subset W$$

So $\|\rho u\|_{H^2(U)} \lesssim \|\tilde\rho u\|_{L^2(W)} + \|\tilde\rho D_x^2 u\|_{L^2(W)}$ for $\tilde\rho = 1$ on supp ρ



$$\|\rho R_0(\lambda)\rho u\|_{H^2(\mathbb{R})} \lesssim \|\tilde\rho u\|_{L^2(\mathbb{R})} + \|\rho u - \tilde\rho \lambda^2 R_0(\lambda)\rho u\|_{L^2(\mathbb{R})} \quad (\text{as } \rho\tilde\rho = \rho,\ D_x^2(R_0(\lambda)u) = u + \lambda^2 R_0(\lambda)u)$$

$$\lesssim |\lambda|^{-1} e^{2L(\mathrm{Im}\lambda)_-}\langle\lambda\rangle^2$$

For other $j \in [0,2]$, interpolate with Hölder's inequality.
This completes the proof.

<u>Interesting and surprising (to me) fact:</u>
This also works for nonzero V!
But requires much more theory without an explicit formula

* <u>Thm (Meromorphic Continuation)</u>
Let $V \in L^\infty_0(\mathbb{R};\mathbb{C})$. Then the resolvent
$R_V = (D_x^2 + V(x) - \lambda^2)^{-1}$ extends to meromorphic family of operators
$R_V(\lambda): L^2_0(\mathbb{R}) \to H^2_{loc}(\mathbb{R})$ with $\lambda \in \mathbb{C}$.
For $\mathrm{Im}\lambda > 0$, we have
$R_V(\lambda): L^2(\mathbb{R}) \to L^2(\mathbb{R})$ with finitely many poles.

<u>Proof Strategy:</u>
1. Construct for $\mathrm{Im}\lambda$ large with Neumann series
2. Construct for $\mathrm{Im}\lambda > 0$ with analytic Fredholm theorem
3. Construct for all λ using cutoff functions
4. Show finitely many poles in upper half plane with bound on R_0.

<u>Proof:</u>
<u>1.</u> Let $\mathrm{Im}\lambda \gg 1$.
Then $R_0(\lambda)$ is "approximate inverse":

$$(P_V - \lambda^2)R_0(\lambda) = (D_x^2 - \lambda^2 + V)R_0(\lambda)$$
$$= I + VR_0(\lambda)$$

But $\|VR_0(\lambda)\|_{L^2 \to L^2} \le \|V\|_{L^\infty}(\mathrm{Im}\lambda)^{-2}$
$\le 1/2$ for $\mathrm{Im}\lambda$ large enough.

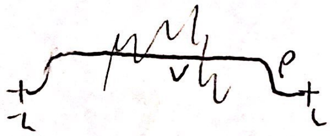
Invert $I + VR_0(\lambda)$ with Neumann series:

$$(I + VR_0(\lambda))^{-1} = \sum_{j=0}^{\infty} (-1)^j (VR_0(\lambda))^j$$

Then $\boxed{R_V(\lambda) = R_0(\lambda)(I + VR_0(\lambda))^{-1}}$

2. Let $\operatorname{Im}\lambda > 0$.

Then pick cutoff function $\rho \in C_0^\infty(-L, L)$, with $\rho = 1$ on $\operatorname{supp} V$



$\rho R_0(\lambda) : L^2(\mathbb{R}) \longrightarrow H_0^2(-L, L)$

This is compact operator on L^2 by Rellich-Kondrachov.

$V\rho R_0(\lambda)$ also compact

$I + VR_0(\lambda) = I + \rho V R_0(\lambda)$ Fredholm

Review of Fredholm Operators

Fredholm = "almost invertible"

P Fredholm $\Leftrightarrow$ $\dim \ker P$ & $\dim \operatorname{coker} P$ are finite

Index is $\operatorname{ind} P = \dim \ker P - \dim \operatorname{coker} P$

Constant on connected sets of Fredholm operators (in norm topology)

*Thm (Fredholm Alternative)

If K is compact, $I + K$ is Fredholm and $\operatorname{ind}(I + K) = 0$

Note about function spaces:

This all works on Banach spaces, but we only need in the case of Hilbert spaces. Also, we only need the $\operatorname{ind} P = 0$ case.

Def: $\underset{\Omega \subseteq \mathbb{C}}{z} \mapsto B(z) \in \mathcal{L}(X, X)$ is holomorphic family $\Leftrightarrow$ $z \mapsto \langle B(z)u, v \rangle$ is holomorphic for all $u, v \in X$

$z \mapsto B(z) \in \mathcal{L}(X, X)$ is meromorphic family $\Leftrightarrow$ For each $z_0 \in \Omega$,

$$B(z) = \underbrace{B_0(z)}_{\text{holomorphic near } z_0} + (z - z_0)^{-1} B_1 + \dots (z - z_0)^{J} B_J$$

finite rank

If for every z_0, B_0 is Fredholm, call B a meromorphic family of Fredholm operators.

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* Thm (Analytic Fredholm Thm)

If $A(z)$ is meromorphic family of Fredholm operators on connected Ω and $A(z_0)^{-1}$ exists for some z_0, then $A(z)^{-1}$ is meromorphic family.

Proof (in holomorphic case & $\underbrace{\text{ind } A(z)=0}_{}$ case)

just a convenience, & all we need anyway

Pick $z_1 \in \Omega$.

If $A(z_1)$ invertible:

$\partial_z(A(z)^{-1}) = -A(z)^{-1}(\partial_z A(z))A(z)^{-1}$ $\quad \left(0 = \partial_z I = \partial_z(AA^{-1}) = (\partial_z A)A^{-1} + A(\partial_z A^{-1}) \right)$

Let $n = \dim \ker A(z) = \operatorname{codim} \operatorname{Ran} A(z)$

Construct Grushin operator $\tilde{A}(z) : X \times \mathbb{C}^n \circlearrowleft$ to be invertible at $z = z_1$

$$\tilde{A}(z) = \begin{pmatrix} A(z) & A_- \\ A_+ & 0 \end{pmatrix} \quad \text{for} \quad \begin{array}{l} A_- : \mathbb{C}^n \to X \\ A_+ : X \to \mathbb{C}^n \end{array}$$

How to construct A_-, A_+?

Let $e_1, \dots e_n$ be basis of $\ker A(z_1)$,

$f_1, \dots f_n$ be basis of $\ker A(z_1)^*$

$$A_+(u) = \begin{pmatrix} \langle u, e_1 \rangle \\ \vdots \\ \langle u, e_n \rangle \end{pmatrix} \qquad A_- \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = v_1 f_1 + \dots + v_n f_n$$

Injectivity:

Let $\binom{u}{u_-} \in X \times \mathbb{C}^n$, $\tilde{A}(z_1)\binom{u}{u_-} = 0$.

$A(z_1)u + A_- u_- = 0$ and $A_+ u = 0$

$A_+ u = 0 \Rightarrow u \perp \ker A(z_1)$

Also $0 = \langle A(z_1)u + A_- u_-, f_j \rangle$
$= \langle A_- u_-, f_j \rangle$

$\Rightarrow u_- = 0$, $A(z_1)u = 0 \Rightarrow u = 0$

Surjectivity:

Let $u = A(z_1)v + w$ for $w \in \ker A(z_1)^*$

$w = \sum a_j f_j$

Then $\tilde{A}(z_1)\begin{pmatrix} v \\ a_1 \\ \vdots \\ a_n \end{pmatrix} = A(z_1)v + w + A_+ v$

Subtract $\langle v, e_1 \rangle e_1 + \dots \langle v, e_n \rangle e_n$ from v to finish

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Then $\tilde{A}(z)$ holomorphic near z_1

$$\tilde{A}(z)^{-1} = \begin{pmatrix} B(z) & B_+(z) \\ B_-(z) & B_\mp(z) \end{pmatrix}$$

Schur Complement Formula: A invertible $\Leftrightarrow$ $B_\mp$ invertible

$A(z)^{-1} = B(z) - B_+(z) B_\mp(z)^{-1} B_-(z)$ $\quad$ (Proof: Just plug in!)

But...

$B_{\mp}$ is meromorphic family of matrices

$\det(B_{\mp})$ is meromorphic function with discrete poles & zeros

$\Rightarrow B_{\mp}(z)^{-1}$ is meromorphic family by Cramer's rule

This gives $A(z)^{-1}$ to be meromorphic family.

Return to $R_V(\lambda)$

We had $I + VR_0(\lambda)$ Fredholm, so

$(I + VR_0(\lambda))^{-1}$ extends to meromorphic family

$R_V(\lambda) = R_0(\lambda)(I + VR_0(\lambda))^{-1} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ extends to $\operatorname{Im}\lambda > 0$.

3. Let $\lambda \in \mathbb{C}$.

$K(\lambda) = VR_0(\lambda) : L^2_0 \to L^2_0$ (technically, $\rho K(\lambda)\rho$ is)

Then $(I + K(\lambda)(1-\rho))^{-1} = I - K(\lambda)(1-\rho)$ (remember $\rho V = V$)

$$(I + K(\lambda))^{-1} = \underbrace{(I + K(\lambda)\rho)^{-1}}(I - K(\lambda)(1-\rho))$$

exists by Neumann series for $\operatorname{Im}\lambda$ large enough

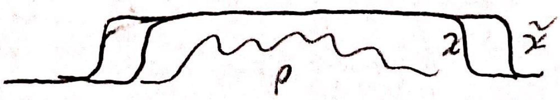
$\therefore R_V(\lambda) = R_0(\lambda)(I + \underbrace{K(\lambda)\rho}_{\text{(compact by R-K)}})^{-1}(I - K(\lambda)(1-\rho))$

Apply analytic Fredholm theorem to $(I + K(\lambda)\rho)^{-1}$

To show correct mapping properties of $(I + K(\lambda)\rho)^{-1}$:

Let $\tilde{\chi} = 1$ on $\operatorname{supp}\chi$

$\chi = 1$ on $\operatorname{supp}\rho$



Then $(1-\tilde{\chi})(I + K(\lambda)\rho)^{-1}\chi = 0$ for $\operatorname{Im}\lambda \gg 1$ by Neumann series

$= 0$ for all λ by analytic continuation

Therefore $(I + K(\lambda)\rho)^{-1} : L^2_0 \to L^2_0$ as desired.

4. Let $\operatorname{Im}\lambda > 0$.

$\|K(\lambda)\rho\|_{L^2 \to L^2} \leq \frac{C}{|\lambda|}$

$R_V(\lambda) = R_0(\lambda)\underbrace{(I + K(\lambda)\rho)}^{-1}(I - K(\lambda)(1-\rho))$

invertible for $|\lambda| > 2C$ $\Longrightarrow$ poles are in $B_{2C}(0)$, so finite.

9) Poles of $R_V(\lambda)$ are called <u>resonances</u>. (or <u>scattering resonances</u>)

The <u>multiplicity</u> is $m_R(\lambda) = \operatorname{rank} \oint_{C_\lambda} R_V(\zeta)\, d\zeta$

<u>Interpretation of Resonances; Solving the Helmholtz Equation</u>

***<u>Thm</u> (<u>Regular Points of R_V</u>):**

If λ is not a resonance and $f \in L^2_0(\mathbb{R})$, the equation

$(P_V - \lambda^2)u = f$

has unique outgoing solution $u = R_V(\lambda)f$.

<u>Proof</u>: <u>Solves Equation</u>: $(P_V - \lambda^2)R_V(\lambda)f = f$

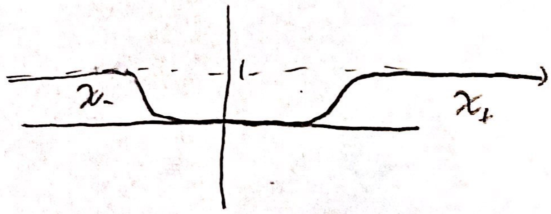
<u>Outgoing Condition</u>: $(\partial_x \mp i\lambda)(R_V(\lambda)f)(\pm R) = 0$ where $\operatorname{supp} f, \operatorname{supp} V \subseteq (-R, R)$

] Hold in $\operatorname{Im}\lambda \gg 1$, so for all λ by analytic continuation

<u>Uniqueness</u>: Claim $R_V(\lambda)(P_V - \lambda^2)u = u$ for outgoing $u \in H^2_{loc}$

True for $\operatorname{Im}\lambda > 0$

For general λ, write $u = \underbrace{u_0}_{\text{in } H^2_0} + \chi_+ a_+ e^{i\lambda x} + \chi_- e^{-i\lambda x}$



Then $u_{\substack{\pm \\ 0}} = R_V(\lambda)(P_V - \lambda^2)u_{\substack{\pm \\ 0}}$ for $\operatorname{Im}\lambda > 0$

$(P_V - \lambda^2)u_{\substack{\pm \\ 0}} \in L^2_0$, so analytic continuation of $R_V(\lambda)$ applies.

***<u>Thm</u> (<u>Singular Points of R_V</u>):**

If $\lambda_0 \neq 0$ is a resonance with $m_R(0) > 0$. Then there is outgoing u_1 with

$(P_V - \lambda_0^2)u_1 = 0$

<u>Remark</u>: In fact there are $u_2, \dots u_{m_R(\lambda_0)}$ with

$(P_V - \lambda_0^2)u_j = u_{j-1}$ for $j \leq m_R(\lambda_0)$

Here u_1 is called a <u>resonant state</u> (and $u_2, \dots$ are <u>generalized resonant states</u>).

<u>Proof for simple resonances</u>

We have $R_V(\lambda) = (\lambda - \lambda_0)^{-1}B_1 + B_0(\lambda)$

$I = (\lambda - \lambda_0)^{-1}(P_V - \lambda_0^2)B_1 + \underbrace{H(\lambda)}_{\text{holomorphic}} \implies (P_V - \lambda_0^2)B_1 = 0$

Pick some ψ with $B_1(\psi) \neq 0$, so

$(P_v - \lambda_0^2)(B_1\psi) = 0$

To show $B_1\psi$ is outgoing:

$(D_x^2 - \lambda^2) R_v(\lambda) = I - VR_v(\lambda)$

$R_v(\lambda) = R_0(\lambda) - R_0(\lambda) V R_v(\lambda)$

Integrate in small circle about λ_0 to get

$B_1 = -R_0(\lambda_0) V B_1$

$B_1\psi = -R_0(\lambda_0) \underbrace{(VB_1\psi)}_{\text{in } L^2_0}$

So $B_1\psi$ outgoing solution.

Where are the resonances?

Consider now real-valued V

*Prop: If λ is resonance and $\operatorname{Im}\lambda > 0$, λ is eigenvalue and $\lambda = ir$ for $r \in \mathbb{R}$.

Proof: We have for x large

$u(x) = a_\pm e^{\pm i\lambda x}$
$\in L^2(\mathbb{R})$

So λ is eigenvalue.

As $P_v = D_x^2 + V$ is self-adjoint, $\lambda^2 \in \mathbb{R}$, $\Rightarrow \lambda = ir$

*Prop: There are no resonances in $\mathbb{R} \setminus \{0\}$

Proof: If $\lambda \in \mathbb{R} \setminus \{0\}$ were resonance, would have outgoing u with

$(P_v - \lambda^2) u = 0 \quad (P_v - \lambda^2)\bar{u} = 0$

Wronskian $W(u, \bar{u}) = u\bar{u}' - u'\bar{u}$

$\partial_x W(u, \bar{u}) = u\bar{u}'' - u''\bar{u}$

$= 0 \qquad \Rightarrow W$ constant.

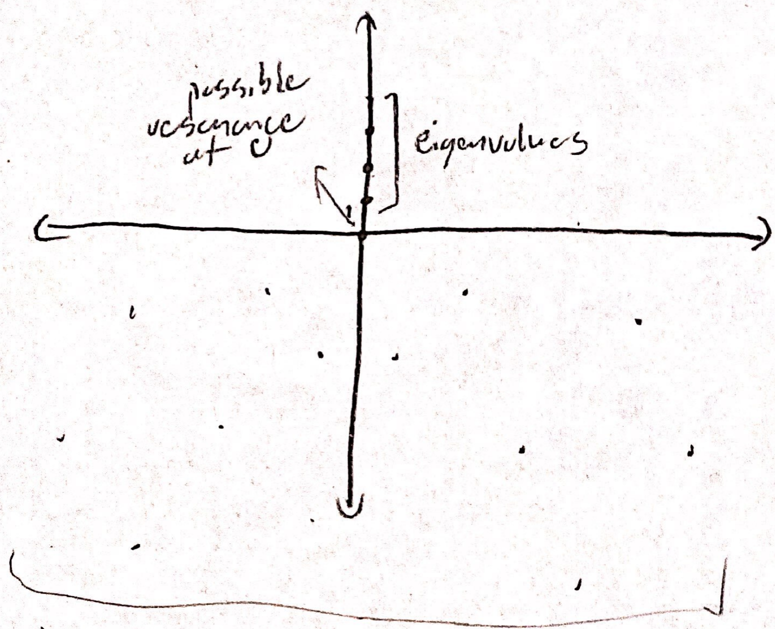
But for $\pm x \gg 1$:

$u(x) = a_\pm e^{\pm i\lambda x} \qquad \bar{u}(x) = \bar{a}_\pm e^{\mp i\lambda x}$

$W(u, \bar{u}) = |a_\pm|^2 (\mp 2i\lambda)$

Gives $|a_+|^2(-2i\lambda) = |a_-|^2(2i\lambda) \Rightarrow |a_+|^2 + |a_-|^2 = 0$, so not a resonant state.

Picture



Don't know anything yet!